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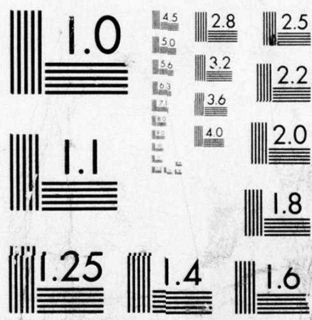

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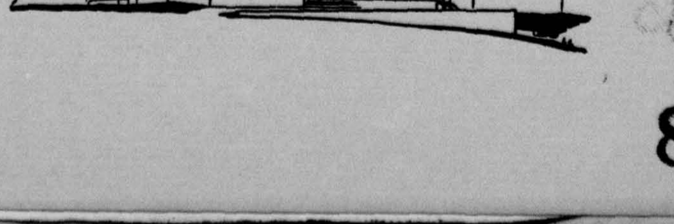

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6 AN EXTENSION OF SAIGAL'S CLASS OF Q-MATRICES

by

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A UNIFICATION OF TWO CLASSES OF Q-MATRICES

Jong-Shi Pang

ABSTRACT. This note presents a class of Q-matrices which includes Saigal's class N of Q-matrices with negative principal minors and the class E of strictly semi-monotone Q-matrices.

Key Words: Class of matrices, Linear complementarity problem.

1. Introduction. Given a real square matrix M and a real vector q of the same size, the linear complementarity problem (q, M) is to find a vector x such that

$$q + Mx \geqq 0 , \quad x \geqq 0 \quad \text{and} \quad x^T(q + Mx) = 0.$$

The matrix M is a Q-matrix if the problem (q, M) has a solution for all vectors q.

The problem of constructively identifying a Q-matrix has yet to be solved. Over the past years, numerous classes of matrices have been shown to belong to this large yet very much unknown class of Q-matrices. Two such classes are discovered by Eaves [2] and Saigal [7]. Eaves' class is $L_1 \cap L_2 \cap S$ where L_1 consists of the semi-monotone matrices M which are square matrices such that for each vector $0 \neq x \geqq 0$, there is an index k such that $x_k > 0$ and $(Mx)_k \geqq 0$; L_2 consists of the square matrices M such that if x is a nonzero solution of the problem (0, M), then there exist nonnegative diagonal matrices D_1 and D_2 with $D_2 x \neq 0$ and $(D_1 M + M^T D_2)\, x = 0$; and S consists of matrices M for which there is a vector $x > 0$ such that $Mx > 0$. Saigal's class is $N \cap S$ where N consists of square matrices M with negative principal minors. These two classes, namely, $L \cap S$ and $N \cap S$ where $L = L_1 \cap L_2$, are in fact distinct because

$$\begin{bmatrix} -1 & 2 \\ 4 & -1 \end{bmatrix} \in (N \cap S) \setminus L_1 \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in (L \cap S) \setminus N .$$

Included among Eaves' class of Q-matrices is the subclass E of strictly semi-monotone matrices which are square matrices M such that for each $0 \neq x \geqq 0$, there is an index k with $x_k > 0$ and $(Mx)_k > 0$. The two matrices just given illustrate that the two classes E and $N \cap S$ are not contained in one another.

Our purpose in the note is to present a class of Q-matrices which properly contains the two classes E and $N \cap S$. The construction of this unifying class is very much motivated by the proof used in [7] to establish that $N \cap S \subseteqq Q$.

2. Main Results. Let A be an n by n matrix and α an index subset of $\{1,\ldots,n\}$. Suppose that the principal submatrix $A_{\alpha\alpha}$ is nonsingular. Let P be a permutation matrix such that P^TAP has $A_{\alpha\alpha}$ as a leading principal submatrix. The α-principal pivot transform of A is defined as the matrix PA^*P^T where

$$(1) \qquad A^* = \begin{pmatrix} A_{\alpha\alpha}^{-1} & -A_{\alpha\alpha}^{-1}A_{\alpha\beta} \\ A_{\beta\alpha}A_{\alpha\alpha}^{-1} & A_{\beta\beta} - A_{\beta\alpha}A_{\alpha\alpha}^{-1}A_{\alpha\beta} \end{pmatrix}.$$

Following the notations of Garcia [4], we let $E^*(d)$ for $d > 0$ or $d = 0$, to denote the class of square matrices M for which the problem (d, M) has zero as the unique solution.

We note that A is a Q-matrix if and only if each of its principal pivot transforms is so, i.e., Q-matrices are invariant under principal pivot transforms. Notice however, that the two classes E and $N \cap S$ are not invariant under such transforms.

We say that an n by n matrix A is an $\tilde{N}$-matrix if there is a vector $f > 0$ and a subset α of $\{1,\ldots,n\}$ satisfying the conditions below

(i) $\quad Af > 0$

(ii) $\quad A_{\alpha\alpha}$ is nonsingular

(iii) For any (n - 1) by (n - 1) principal submatrix $\tilde{A}$ of the α-principal pivot transform of A, it holds that $\tilde{A} \in E^*(\tilde{f}) \cap E^*(0)$ where $\tilde{f}$ is the (n - 1)-

subvector of the (positive) vector $\bar{f}$ which in partitioned form (according to A* given in (1)) is defined as

$$\bar{f} = \begin{pmatrix} f_\alpha \\ A_{\beta\alpha} f_\alpha + A_{\beta\beta} f_\beta \end{pmatrix}.$$

The components of $\tilde{f}$ are in correspondence with the rows of $\tilde{A}$

In the above definition of an $\tilde{N}$-matrix, we allow $\alpha = \phi$. Of course, the ϕ-principal pivot transform of A is A itself. Note that condition (i) implies that an $\tilde{N}$-matrix is necessarily an S-matrix. According to [4], the principal submatrix $\tilde{A}$ is a Q-matrix. The result below shows that an $\tilde{N}$-matrix is in fact a Q-matrix.

<u>Theorem 1</u>. An $\tilde{N}$-matrix is a Q-matrix.

<u>Proof</u>. Let A be a $\tilde{N}$-matrix and let A* be the matrix given in (1). Let q be a given vector and let

$$q^* = \begin{pmatrix} -A_{\alpha\alpha}^{-1} q_\alpha \\ q_\beta - A_{\beta\alpha} A_{\alpha\alpha}^{-1} q_\alpha \end{pmatrix}.$$

It suffices to show that the linear complementarity problem (q^*, A^*) has a solution. We may write this latter problem as

$$x_\alpha = -A_{\alpha\alpha}^{-1} q_\alpha + A_{\alpha\alpha}^{-1} u_\alpha - A_{\alpha\alpha}^{-1} A_{\alpha\beta}\, x_\beta \geqq 0, \qquad u_\alpha \geqq 0$$

$$u_\beta = q_\beta - A_{\beta\alpha} A_{\alpha\alpha}^{-1} q_\alpha + A_{\beta\alpha} A_{\alpha\alpha}^{-1} u_\alpha + (A_{\beta\beta} - A_{\beta\alpha} A_{\alpha\alpha}^{-1} A_{\alpha\beta})\, x_\beta \geqq 0, \quad x_\beta \geqq 0$$

$$(x_\alpha)^T u_\alpha = (x_\beta)^T u_\beta = 0.$$

Consider the solution of the problem by Lemke's almost complementary

pivoting algorithm [6] using $\bar{f}$ defined in condition (iii) above as the artificial vector. If at some point in the solution process, both x_α and u_β become nonbasic, then the system below has a solution:

$$0 = -A_{\alpha\alpha}^{-1} q_\alpha + \lambda f_\alpha + A_{\alpha\alpha}^{-1} u_\alpha \quad - A_{\alpha\alpha}^{-1} A_{\alpha\beta} x_\beta, \qquad u_\alpha \geq 0, \qquad \lambda \geq 0,\ x_\beta \geq 0$$

$$0 = q_\beta - A_{\beta\alpha} A_{\alpha\alpha}^{-1} q_\alpha + \lambda(A_{\beta\alpha} f_\alpha + A_{\beta\beta} f_\beta) + A_{\beta\alpha} A_{\alpha\alpha}^{-1} u_\alpha + (A_{\beta\beta} - A_{\beta\alpha} A_{\alpha\alpha}^{-1} A_{\alpha\beta}) x_\beta .$$

This latter system is clearly equivalent to the one

$$0 = q_\beta + A_{\beta\beta}(x_\beta + \lambda f_\beta), \qquad x_\beta \geq 0, \qquad \lambda \geq 0$$

$$u_\alpha = q_\alpha - \lambda(A_{\alpha\alpha} f_\alpha + A_{\alpha\beta} f_\beta) + A_{\alpha\beta}(x_\beta + \lambda f_\beta) \geq 0 .$$

The consistency of the last system implies that the one below is solvable

$$0 = q_\beta + A_{\beta\beta}\, \bar{x}_\beta, \qquad \bar{x}_\beta \geq 0$$

$$\bar{u}_\alpha = q_\alpha + A_{\alpha\beta}\, \bar{x}_\beta \geq 0.$$

In fact, we have $\bar{x}_\beta = x_\beta + \lambda f_\beta$ and $\bar{u}_\alpha = u_\alpha + \lambda(A_{\alpha\alpha} f_\alpha + A_{\alpha\beta} f_\beta)$. Therefore in this case, the problem (q, A) has a solution. So suppose that throughout the solution of the problem (q^*, A^*) by Lemke's algorithm (with the above choice of artificial vector $\bar{f}$), at least one variable in $\begin{pmatrix} x_\alpha \\ u_\beta \end{pmatrix}$ is basic. If the algorithm terminates in a ray, then the problem ($\bar{\lambda}\bar{f}$, A^*) for some nonnegative $\bar{\lambda}$, has a nonzero solution. This implies by the fact that at least one variable in $\begin{pmatrix} u_\alpha \\ x_\beta \end{pmatrix}$ must be zero, that a certain principal subproblem ($\tilde{f}$, $\tilde{A}$) (in the case $\bar{\lambda} > 0$) or (0, $\tilde{A}$) (in the case $\bar{\lambda} = 0$) where $\tilde{A}$ is an (n - 1) principal submatrix of A^* and $\tilde{f}$ the corresponding (n - 1)-subvector of $\bar{f}$, would have a nonzero solution. But this contradicts

condition (iii). Consequently, Lemke's algorithm must compute a solution of (q^*, A^*). This completes the proof of the theorem.

The proof of the theorem is based on an extension of the argument used in Saigal [7] for the special case $N \cap S$. As a matter of fact, the proof also suggests a constructive method for actually computing a solution to (q, A) with A an $\tilde{N}$-matrix, provided that the index set α and vector f are available readily. Indeed, one can apply Lemke's algorithm to the problem (q^*, A^*) using $\bar{f}$ as the artificial vector. As soon as the artificial variable λ reaches zero or all the u_β and x_α variables become nonbasic, a solution to the given problem (q, A) can be obtained easily.

If a square matrix A is such that some α-principal pivot transform is strictly semi-monotone, then A is an $\tilde{N}$-matrix. This follows from the fact that principal submatrices of strictly semi-monotone matrices are themselves strictly semi-monotone and that a strictly semi-monotone matrix must be in $E^*(d) \cap E^*(0)$ for any positive d (see [2] e.g.). Hence in particular, if A is itself strictly semi-monotone, then A is an $\tilde{N}$-matrix.

On the other hand, if a square S-matrix A is such that some α-principal pivot transform $\bar{A}$ has all proper principal minors positive, then A is an $\tilde{N}$-matrix. This is because any (n-1) by (n-1) principal submatrix of $\bar{A}$ must then be a P-matrix, i.e. has all principal minors positive, and thus belong to $E^*(d) \cap E^*(0)$ for any positive d (see [8]). Hence, in particular, if A is in $N \cap S$ then A is an $\tilde{N}$-matrix. This follows from the fact that A^{-1} which is the $\{1,\ldots,n\}$-principal pivot transform of A, has all proper principal minors positive (see [7]).

That the class of $\tilde{N}$-matrices properly contains the union $E \cup (N \cap S)$ can be seen from the example

$$\begin{bmatrix} -1 & 2 \\ -4 & 7 \end{bmatrix}$$

which is an $\tilde{N}$-matrix (its $\{2\}$-principal pivot transform is a positive matrix) but certainly not strictly semi-monotone or has negative principal minors.

In [5], it is shown that the linear complementarity problem (q, A) with $A \in N$ has 0, 1, 2 or 3 solutions. The theorem below extends this result . Recall that a square matrix is nondegenerate if all its principal minors are nonzero.

<u>Theorem 2</u>. Let A be a nondegenerate matrix such that some α-principal pivot transform $\bar{A}$ has all proper principal minors positive. Then for every vector q, the linear complementarity problem (q, A) has 0, 1, 2, or 3 solutions.

<u>Proof</u>. If $\bar{A}$ has positive determinent, then $\bar{A}$ and thus A is a P-matrix. Hence the problem (q, A) has a unique solution for all vectors q. On the other hand, if $\bar{A}$ does not have positive determinent, then it must have negative determinent. This is because $\bar{A}$ must be nonsingular. In fact, its inverse is a principal rearrangement of the β-principal pivot transform of A with β the complement of α. This latter principal pivot transform is well-defined by the nondegeneracy of A. Consequently, it follows that the inverse of $\bar{A}$ is in class N. Hence, by the result established in [5], the linear complementarity problem $(\bar{q}, \bar{A}^{-1})$ has 0, 1, 2 or 3 solutions. As $\bar{A}^{-1}$ is also a principal pivot transform of A, the same conclusion is true for each (q, A). This completes the proof of the theorem.

We gave an example earlier to show that there are matrices in $N \cap S$ which are not in the class E. The following result establishes that the inverse of a matrix in $N \cap S$ in fact belongs to E.

Theorem 3. Let A be in $N \cap S$. Then the inverse of A is in E.

Proof. In fact, if A is in $N \cap S$, then each proper principal submatrix of A^{-1} is a P-matrix. In particular, each (n - 1) by (n - 1) principal submatrix of A^{-1} is strictly semi-monotone (see [3]). As A^{-1} is also an S-matrix, the desired conclusion now follows from a result established in [1].

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This note presents a class of Q-matrices which includes Saigal's class of Q-matrices with negative principal minors and the class of semi-monotone Q-matrices.